

Superadditivity of distillable entanglement From Quantum Teleportation

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We show that the phenomenon of superadditivity of distillable entanglement observed in multipartite quantum systems results from the consideration of states created during the execution of the standard end-to-end quantum teleportation protocol (and a few additional local operations and classical communication (LOCC) steps) on a linear chain of singlets. Some of these intermediate states are tensor products of bound-entangled (BE) states, and hence, by construction possess distillable entanglement, which can be unlocked by simply completing the rest of the LOCC operations required by the underlying teleportation protocol. We use this systematic approach to construct both new and known examples of superactivation of bound entanglement, and first examples of activation of BE states using other BE states. A surprising outcome is the construction of noiseless quantum relay channels with no distillable entanglement between any two parties, except for that between the two end nodes.

Quantum entanglement is fundamental to quantum mechanics, and has been established as the critical resource for quantum information processing (QIP) and quantum computing. In spite of much recent progress, spurred by the growth of the new field of QIP, a comprehensive functional and conceptual understanding of entanglement continues to remain elusive. In this letter, we explore one such intriguing issue involving the emergence of distillable entanglement in a multi-party quantum system, when none of the constituent subsystems possesses any distillable entanglement.

More precisely, an entangled state is said to be distillable if one can obtain some pure entanglement in an asymptotic sense by LOCC [1]. This class of entangled states can be used to set up quantum channels and provide the basic infrastructure for communication and computation. Even though most entangled states are distillable, there exist entangled states that do not allow distillation of maximally entangled states by LOCC. Such states are known as bound entangled (BE) states [2]. A multipartite entangled state is said to be bound entangled if there is no distillable entanglement between any subset of the parties *as long as all the parties remain separated from each other*. Recent results have shown that bound entangled states in multipartite systems come in two varieties: the activable (or unlockable) and non-activable states. For activable BE (ABE) states [4, 5], if some of the parties group together and perform collective LOCC, then they can distill entanglement between a subset of spatially separated parties. However, not all multipartite bound entangled states are activable, and there exist states where there is no distillable entanglement across

any bipartite partition. An example of such a state is the bound entangled state constructed from a multipartite unextendible product basis [6].

An intriguing example of the richness of entanglement manipulation in a multipartite scenario is the so-called superactivation process, where two activable bound entangled states tensored together produces distillable entanglement between two parties belonging to the two *different* states [7]. It demonstrates superadditivity of distillable entanglement as the individual BE states are not distillable. At this point it is useful to make a *distinction* between *superactivation* and *activation*. In activation, one is provided with a multipartite BE state such that if some of the parties are provided with *additional states* to share, then a subset of the spatially separated parties can distill entanglement among them. In the literature, the shared auxiliary states have been singlets [4, 5], e.g., in an ABE state, bringing a subset of the parties together is equivalent to providing them with singlets. Moreover, in certain multipartite activation processes, the provided singlet is entangled across a different cut than the entanglement obtained in activation, and as such, this information should be always explicitly stated. An important point to note is that all the activation processes *demonstrated show far* correspond to the *subadditive property* of distillable entanglement: In order to distill one maximally entangled state from the given state, one, on the average, has to spend more than one maximally entangled states. One of the results we show is that one can also unlock certain ABE states by providing another BE state, instead of singlets; thus, it is possible to *demonstrate superadditivity of distillable entanglement in the context of the activation process* as well.

What is the physics behind the superadditivity of distillable entanglement in a multipartite situation? To answer this, it is useful to consider the particular four-party

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ABE state that lies at the center of the superactivation process [5, 7]:

$$\rho_s^{ABCD} = \frac{1}{4} \sum_{i=1}^4 (|\Phi_i\rangle \langle \Phi_i|)^{AB} \otimes (|\Phi_i\rangle \langle \Phi_i|)^{CD}, \quad (1)$$

where for convenience sake, we have adopted the following notation for the four Bell states: $\Phi = \{\Psi^+, \Psi^-, \Phi^+, \Phi^-\}$ with elements Φ_i being the maximally entangled states for two qubits (Bell states) and are given by: $|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$, $|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$. The four-party ABE state, ρ_s , has several interesting properties (for details see Ref [5]): it is symmetric under interchange of parties, i.e., $\rho_s^{ABCD} = \rho_s^{ACBD} = \rho_s^{ADCB}$ etc., and any two parties coming together can distill a singlet between the other two which implies the state must be entangled. On the other hand since the state is separable across every 2:2 bipartite cut every party is separated from every other by a separable partition and hence no entanglement can be distilled when all four parties remain spatially separated. Hence the state is bound entangled.

We first show how to optimally prepare the state ρ_s^{ABCD} and then show how it can efficiently replace a pair of singlets in a relay quantum channel; *this ability of the ABE state to replace singlets in a chain*, is the key to the phenomenon of superadditivity of distillable entanglement.

Optimal Preparation of the ABE state ρ_s^{ABCD} in Eq. (1)

Recently it was shown that the Smolin state belong to the family of activable bound entangled states for even number (greater than or equal to four) of qubits [8] and the exact cost of preparation of such states have been shown to be N ebits for a $2N$ qubit ABE state [9]. In particular it was shown that *two ebits are both necessary and sufficient* to prepare the four qubit Smolin state [9]. We now briefly review the sufficiency part of the proof following [9]. The sufficiency part of the proof involves a protocol that utilizes a pair of singlets (hence, requiring two ebits) and LOCC. In fact throughout this paper we will make use of this local protocol. Let the pairs, (A, B), and (C, D), share a singlet each. A and C can classically communicate among themselves to prepare a state $|\Phi_i\rangle^{AA} \otimes |\Phi_i\rangle^{BB}$ randomly with equal probability. A and C then can each teleport one qubit of the correlated Bell states (keeping one qubit from each state to themselves) to B and D using the shared singlets. This creates the ABE state ρ_s^{ABCD} (see Fig. 1).

The construction of the state makes it clear why the ABE state ρ_s^{ABCD} can replace a pair of singlets in a chain. As illustrated in Fig. 1(ii), consider the case where parties A, B, C, and D are linked by three singlets. It is obvious that one can generate a singlet state between the end nodes A and D just by

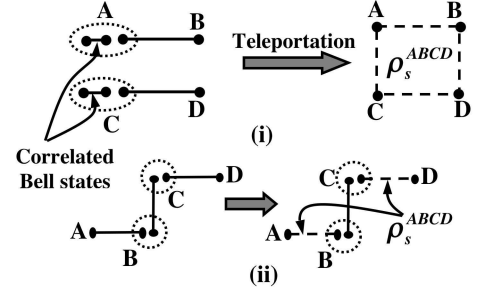


FIG. 1: (i) Optimal preparation of the ABE state ρ_s^{ABCD} . Singlets and other Bell states are shown by solid lines connecting filled circles. Multipartite states are represented by dotted lines connecting the underlying parties. (ii) Teleportation in a chain of singlets and in a system consisting of one ABE state and a singlet.

following the standard teleportation protocol in the intermediate nodes via the process of entanglement swapping. Now suppose the four parties conspire and replace the pair of singlets AB and CD with the ABE state ρ_s^{ABCD} . The state of the quantum channel is now $\rho_s^{ABCD} \otimes (|\Psi^-\rangle \langle \Psi^-|)^{BC}$. Next, *if this modified state is given to four new parties*, then they can treat the chain as if it were still singlets, and distill one ebit of entanglement by executing the *standard teleportation* protocol. To see this, consider the situation where A receives a one-qubit state ρ . The teleportation from A to B happens via an unknown Bell state, and hence the teleported state in B is $\sigma_i \rho \sigma_i^\dagger$, for some unknown Pauli operator σ_i . The state remains the same in C after teleportation from B to C as the channel is a singlet. However in the last step of teleportation from C to D, it takes place using the same but unknown Bell state as that between A and B. Hence, the final state becomes $\sigma_i^2 \rho \sigma_i^{2\dagger}$; as $\sigma_i^2 = I$, the final state at node D is again ρ .

Remark 1: These arguments also show why ρ_s^{ABCD} is *activable*: providing a singlet between B and C is the same as bringing the pair together; hence, by bringing B and C together we can distill an ebit between A and D. In other words, there is one ebit of distillable entanglement across any 1:3 cut in ρ_s^{ABCD} .

Remark 2: While we have presented the case of a chain with three singlets, the arguments easily generalize to the case of chains of singlets of any length: *one can distill one ebit of entanglement between the two end parties*, by performing the *standard teleportation* protocol, even if any number of *pairs of singlets* are first converted to the ABE state ρ_s^{ABCD} .

The properties of the state ρ_s^{ABCD} are, however, indeed different from that of a pair of singlets. To see this, consider the two qubit channels formed by a pair of singlets, and by the four-party state ρ_s^{ABCD} . Given any 2-qubit state ρ at one end of the channel, the tele-

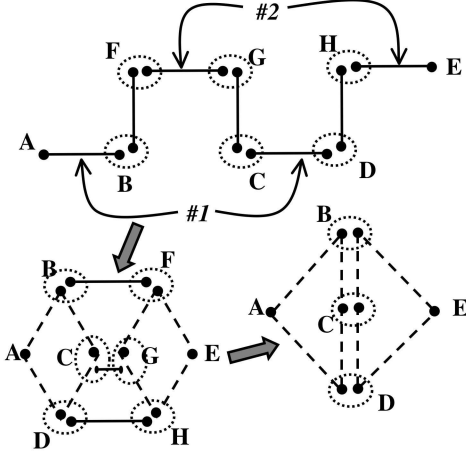


FIG. 2: Superactivation involving two four-party activable bound-entangled states (ABEs), as constructed from a singlet chain via LOCC and the standard teleportation protocol.

ported state on the other side is $\frac{1}{4} \sum_{i=1}^4 (\sigma_i \otimes \sigma_i) \rho(\sigma_i \otimes \sigma_i)$, which will be same as the input state, ρ , only for special states. For example 2-qubit states that are invariant under $\sigma_i \otimes \sigma_i$ operations, can be transmitted exactly over the ABE channel. For instance, Werner states are indeed invariant under such an operation and hence, they can be teleported unaltered via the 2-qubit channel ρ_s^{ABCD} .

Superactivation from Teleportation

Consider the chain of seven singlets, as shown in Fig. 2. As the figure illustrates, two pairs of singlets can be replaced with their corresponding four-party ABE states, ρ_s . It follows from Remark 2 that the resulting state has one ebit of entanglement between A and E, which can be recovered by the standard teleportation protocol. However, as shown in the figure, if one uses the remaining three singlets to bring the respective parties together, then the chain reduces to the tensor product state: $\rho_s^{ABCD} \otimes \rho_s^{EFGH}$. This is the case of *superactivation* introduced in [7]: the tensor product of two ABE states leads to distillable entanglement between two parties belonging to the two different states.

As shown in Fig. 3, one could replace a third pair (e.g., the singlets BF and DH) in the original 7-singlet chain by the corresponding ABE state. It follows again from Remark 2 that the resulting state still has one ebit of entanglement between A and E, which can be distilled by the standard teleportation protocol. Next, if one brings C and G together by using the singlet CG, then one gets a *new superactivation scenario*: The tensor product state, $\rho_s^{ABCD} \otimes \rho_s^{BDFH} \otimes \rho_s^{CEFH}$, of three ABE states leads to one ebit of distillable entanglement between the nodes A and E. Clearly, one can now create infinitely many such superactivation configurations.

Remark 3: In the original superactivation configuration, as illustrated in Fig. 2, the state is the tensor product of two *disjoint* ABE states connected via

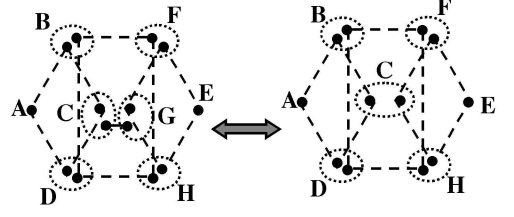


FIG. 3: Superactivation involving three ABE states.

three singlets: $\rho_s^{ABCD} \otimes \rho_s^{EFGH} \otimes (|\Psi^-\rangle \langle \Psi^-|)^{BF} \otimes (|\Psi^-\rangle \langle \Psi^-|)^{CG} \otimes (|\Psi^-\rangle \langle \Psi^-|)^{DH}$. Can the two ABE states be activated with less than three singlets, or equivalently, by sharing less than three parties between the two ABE states? Going back to the original 7-singlet chain, one can easily show that (i) the original chain will break up into at least two pieces, if one or more of the three singlets are removed, and (ii) each connected chain will have only one of the two correlated Bell states from at least one of the ABE states. For example, if the singlet BF is removed in Fig. 2, then the chain breaks into two, and the two pairs, AB and CD, from the state ρ_s^{ABCD} , are in different connected chains. Because of these unmatched correlated Bell states, each chain acts as a depolarizing channel; distillable entanglement is obtained only if both the pairs of the ABE state ρ_s are in the same connected chain. Hence, two ABE states, ρ_s , must share three parties in order to have distillable entanglement between A and E. In other words, *any state comprising two such states ρ_s is a ABE state if only two or less parties are shared between the two states.*

Superadditivity of Distillable Entanglement in the Activation Scenario

Consider the superactivation configuration involving three ABE states: $\rho_s^{ABCD} \otimes \rho_s^{BDFH} \otimes \rho_s^{CEFH}$ (see Fig. 3). As already mentioned, in order to distill one ebit of entanglement between A and E, the rest of the nodes can follow the standard teleportation protocol, where the nodes can perform their respective Bell measurements and announce the results to one of their neighbors *in any order* (follows from Remark 2 and the commuting properties of Bell measurements). Consider the case where the node C performs its Bell measurement (BM) and announces its results. After this, the state becomes the tensor product of one six-party state and one four-party state: $\rho_x^{ABDEFH} \otimes \rho_s^{BDFH}$. Since one can still distill one ebit of entanglement between A and E by having the rest of the nodes complete their BM's and classical communications (CC's), the new configuration, $\rho_x^{ABDEFH} \otimes \rho_s^{BDFH}$, is distillable.

We have already shown in Remark 3 that the configuration $\rho_s^{ABCD} \otimes \rho_s^{CEFH}$ (i.e., two ABE states ρ_s , where only one party is shared between the two) is ABE, and the state ρ_x^{ABDEFH} is obtained from the two states via LOCC; hence, *the state ρ_x^{ABDEFH} is also a ABE state.*

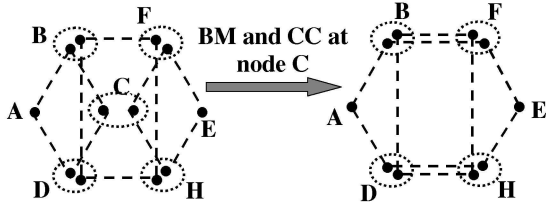


FIG. 4: Superadditivity of distillable entanglement in the activation framework.

Thus, the distillable state $\rho_x^{ABDEFH} \otimes \rho_s^{BDFH}$ can be viewed as follows: One is given a six-party BE state ρ_x^{ABDEFH} , which is unlocked by using another BE state, ρ_s^{BDFH} , as an auxiliary resource. Thus the BE state ρ_x^{ABDEFH} is activated, not by bringing parties together or by giving free entanglement, but by another BE state. This constitutes a case of superadditivity of distillable entanglement in the activation scenario. Note that we obtained the activation case from the superactivation case by performing one of the BM's necessary in the distillation process. This reiterates our thesis that the superadditivity phenomenon in multipartite systems is just a manifestation of the distillation process via teleportation.

Discussions

We have shown that all the known and several new instances of the phenomena of superadditivity of distillable entanglement in multipartite quantum systems can be systematically derived by performing the *standard quantum teleportation or entanglement-swapping protocol* in a chain of singlets. One might ask if the same mechanism would hold for *any* case of superactivation that might be discovered in the future. It is a well-known truism in QIP that given enough pairwise ebits, *any* multipartite state can be prepared via LOCC. For example, if one party, say A, shares one ebit with every other party then this configuration guarantees preparation of any multipartite state; such a configuration is referred to as a *star network*, with A as the central hub. This is because, A can prepare the multipartite state, and teleport corresponding qubits to each of the other parties. Equivalently, starting from a chain of sufficient number of ebits, the above star configuration can always be generated by LOCC. Next, one can always replace the ebits by sufficient numbers of singlets, leading to the conclusion that a nonuniform chain of singlets (where we allow neighboring nodes to share more than one singlet) can generate any superactivation configuration via LOCC. Thus, a nonuniform *chain of singlets* is a *universal configuration*, and can lead to any superactivation mechanism. The appealing aspects of the results in this paper are that (i) the chain of singlets is uniform and has exactly one singlet between neighboring nodes, and (ii) after the initial LOCC operations to set up the ABE states, the superactivation configurations can be constructed by exclusively executing the

standard teleportation protocol. This provides the *first physical explanation* and a constructive procedure for a phenomenon that was originally presented and perceived by the community as a puzzling and surprising aspect of quantum entanglement.

We end the communication by considering a rather simple question: Suppose we want a quantum relay channel connecting nodes A and E, and going through nodes B, C, and D. Is it necessary that the pairs of neighboring nodes, (A,B), (B,C), (C,D), and (D,E) must have distillable entanglement? Quite unexpectedly, we find that the answer is in the negative. Consider a channel comprising a linear chain of four singlets; clearly, the neighboring nodes in the chain are maximally entangled. Now, however, we use the singlets, AB and CD, to prepare (via LOCC) the ABE state ρ_s^{ABCD} , and the singlets, BC and DE, to prepare the ABE state ρ_s^{BCDE} . This leads to the superactivation configuration, $\rho_s^{ABCD} \otimes \rho_s^{BCDE}$, from which one can distill one ebit of entanglement between A and E. Thus, the state $\rho_s^{ABCD} \otimes \rho_s^{BCDE}$ comprises a noise-less quantum relay channel, but with a *surprising twist*: if one considers any pair of nodes in the channel (including, the pairs formed by the neighboring nodes), except the end pair (A, E), then there is no distillable entanglement between the parties in the pair!

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